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Representing Spence's Integral by Elementary Functions

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Abstract—An interesting connection exists between Spence's integral, used in Feynman diagrams in particle physics, and the variance of the reciprocal of a geometric random variable, used in probability theory. This linkage leads to approximate representations for Spence's integral over the unit interval which works well in practice. The result shows how the interplay between probability and physics can bear pragmatic fruit.

Keywords—Approximation, Dilogarithm function, Spence's integral, Feynman diagrams, Particle physics, Applied probability, Differential equations.

Science is as much for intellectual enjoyment as for practical utility.
—Richard P. Feynman

INTRODUCTION

Spence's integral plays a role in Feynman diagrams in elementary particle physics [1,2]. It defines a special function known as the Dilogarithm function through the following equation:

$$\text{Polylog}\,(2, z) = \int_z^0 \left(\frac{1}{t}\right) \log\,(1-t)\,dt.$$

An unexpected connection exists between this special function and the reciprocal of the geometric probability distribution which frequently arises in applied statistical work. This linkage is exploited to derive approximate representations of Spence's integral over the unit interval $(0, 1)$. The approximations are very accurate for values of z in the interval $(0, 0.73)$. These results have been useful in our applications and may be of value to other researchers working with Spence's integral.

RESULTS

Proposition 1. *Let X be a geometric random variable with parameter p. Then the following results hold:*

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$$E\left[\frac{1}{X}\right] = \frac{-p\log[p]}{1-p},$$

$$\mathrm{Var}\left[\frac{1}{X}\right] = p\,\mathrm{Polylog}\,\frac{(2,1-p)}{1-p} - \left(\frac{p\log[p]}{1-p}\right)^2,$$

where Var *denotes the variance of the random variable.*

PROOF. Let $X \sim$ geometric$[p]$, so that its probability law is

$$f(x) = q^{x-1}p, \qquad x = 1,2,\ldots,$$

where $q = 1-p$ (see [3]). Then, $E[1/X]$ is given by

$$E\left[\frac{1}{X}\right] = f(x) = q^{x-1}p = pq^{x-1} = p\left(1+\frac{q}{2}+\frac{q^2}{3}+\frac{q^3}{4}+\cdots\right)$$
$$= \left(\frac{p}{q}\right)\left(q+\frac{q^2}{2}+\frac{q^3}{3}+\cdots\right) = -\left(\frac{p}{q}\right)\log(1-q) = -\frac{p\log[p]}{1-p}.$$

Next, note that $E[1/X^2]$ is given by

$$E\left[\frac{1}{X^2}\right] = f(x) = q^{x-1}p = pq^{x-1} = p\left(1+\frac{q}{2^2}+\frac{q^2}{3^2}+\frac{q^3}{4^2}+\cdots\right).$$

To sum the above infinite series, we derive a differential equation whose solution is the sum. Let

$$S = 1+\frac{q}{2^2}+\frac{q^2}{3^2}+\frac{q^3}{4^2}+\cdots.$$

Then

$$qS = q+\frac{q^2}{2^2}+\frac{q^3}{3^2}+\frac{q^4}{4^2}+\cdots.$$

Since every power series can be differentiated term by term within its radius of convergence,

$$\frac{d[qS]}{dq} = 1+\frac{2q}{2^2}+\frac{3q^2}{3^2}+\frac{4q^3}{4^2}+\cdots = 1+\frac{q}{2}+\frac{q^2}{3}+\frac{q^3}{4}+\cdots$$
$$= \left(\frac{1}{q}\right)\left(q+\frac{q^2}{2}+\frac{q^3}{3}+\frac{q^4}{4}+\cdots\right) = -\left(\frac{1}{q}\right)\log(1-q).$$

Solving the differential equation for q yields

$$qS = \mathrm{Polylog}\,(2,q)$$

where the special function Polylog$\,(n,z)$ is given by

$$\mathrm{Polylog}\,(n,z) = \frac{z^k}{k^n}.$$

Our solution follows by noting that the dilogarithm Polylog$\,(2,z)$ satisfies

$$\mathrm{Polylog}\,(2,z) = \int_z^0 \left(\frac{1}{t}\right)\log(1-t)\,dt.$$

Note that qS and Polylog$\,(2,q)$ are both zero when q is zero. When $q \neq 0$, we obtain

$$S = \mathrm{Polylog}\,\frac{(2,q)}{q} = \mathrm{Polylog}\,\frac{(2,1-p)}{1-p}.$$

Since $\mathrm{Var}[1/X] = E[1/X]^2 - (E[1/X])^2$, we get

$$\mathrm{Var}\left[\frac{1}{X}\right] = p\,\mathrm{Polylog}\,\frac{(2,1-p)}{1-p} - \left(\frac{p\log[p]}{1-p}\right)^2. \qquad \blacksquare$$

PROPOSITION 2. *Spence's integral may be approximated by the following representation in terms of linear, quadratic, hyperbolic and logarithmic functions, valid over the interval* $(0,1)$,

$$\text{Polylog }(2,p) \sim R_c(p) = (1-p)\frac{\{\log(1-p)\}^2}{p} + \frac{(1-p)\{(1+\log(1-p))/p\}^2}{c}$$

where c is the correlation between X and $1/X$.

PROOF. The covariance between the random variables X and $1/X$ is given by $\text{Cov}[X,1/X] = E[X-(1/X)] - E[X]E[1/X] = 1 - E[X]E[1/X] = 1 + \log[p]/(1-p)$, where we have used Proposition 1 for $E[1/X]$ and the well-known result $E[X] = 1/p$. Next, the correlation between X and $1/X$ is defined as $\text{Corr}[X,1/X] = \text{Cov}[X,1/X]/\{\text{Var}[X]\text{Var}[1/X]\}^{0.5} = \{1+\log(p)/(1-p)\}/\{D(p)\}^{0.5}$, where $D(p) = (1-p)\left(p\,\text{Polylog}(2,1-p)/(1-p) - (p\log[p]/(1-p))^2\right)/p^2$. The correlation between X and $1/X$ will be negative and ≤ 1 in absolute magnitude. If we approximate the correlation between X and $1/X$ by the parameter c, and replace p by $1-p$, then we obtain the representation $R_c(p)$. Approximating the correlation c by its upper bound, squaring both sides of the equation for the correlation and replacing p by $1-p$ results in one possible representation for Polylog $(2,p)$. Another representation can be derived by approximating the correlation by its average value over the interval $(0,1)$. Other choices for c are possible, but these seem to be the most natural ones.

QUALITY OF APPROXIMATE REPRESENTATIONS

One way to examine the quality of the approximation is to analyze the mean square error from the ordinary least squares regression of $\text{Polylog}(2,p)$ on its approximate representation $R_c(p)$. I find that the mean square error is less than 5% for $p \,\varepsilon (0,1)$. However, the mean square error is an *average* across the entire range of p. Although the average error may be small over the entire range, the approximation error may still be large at some values of p. In practice, it would be far more useful to know the approximation error for any given value of p. The approximation error curves reproduced below show how the error varies over the entire range of p. These curves correspond to $R_1(p)$ (the representation obtained by approximating c by the maximum correlation 1) and $R_{\text{av}}(p)$ (the representation obtained by approximating c by its average value over $(0,1)$). I computed the average value of $\text{Corr}[X,1/X]$ in two ways to check the accuracy of the result. Dividing $(0,1)$ into 100 equal subintervals and computing the average value of $\text{Corr}[X,1/X]$ results in the value -0.8154. Dividing $(0,1)$ into infinitesimally small intervals improves the accuracy but requires computing $\int_0^1 \text{Corr}[X,1/X]\,dp$, which is analytically infeasible, given the nonlinearity of the integrand. Using a numerical integration package and computing $\int_0^1 \text{Corr}[X,1/X]\,dp$ yields the value -0.813 for the average correlation. The numerical integration was accomplished by a Gaussian quadrature rule implemented on the symbolic manipulation package Mathematica (see Figure 1).

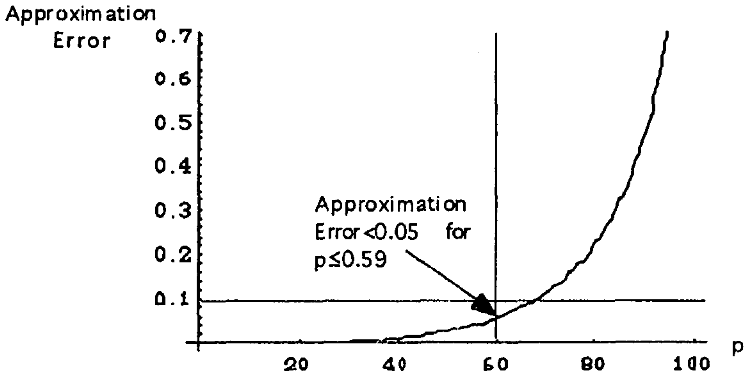
Tables 1 and 2 summarize, for each representation, the range of values of p over which the approximation error is between 1 and 10%. These are the typical percentage errors deemed acceptable in many applications.

Table 1. Representation corresponding to upper bound.

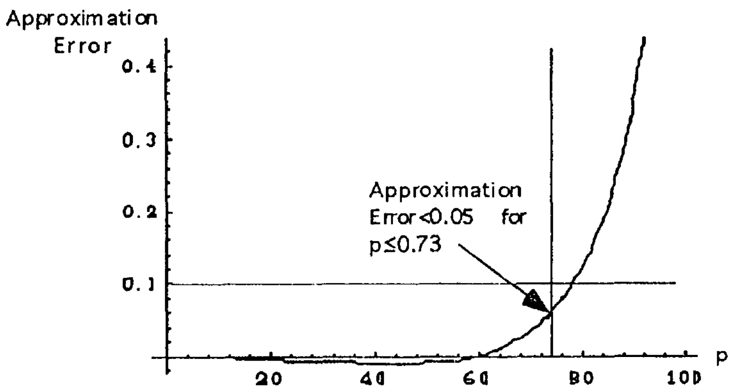
For p less than	0.39	0.47	0.52	0.56	0.59	0.61	0.64	0.65	0.67	0.69
Percent error less than	1	2	3	4	5	6	7	8	9	10

Table 2. Representation corresponding to mean correlation.

For p less than	0.64	0.67	0.70	0.72	0.73	0.75	0.76	0.77	0.78	0.79
Percent error less than	1	2	3	4	5	6	7	8	9	10



(a) Approximating the squared correlation by its upper bound: $c = 1$. $R_1(p) = (1-p)\{\log(1-p)\}^2/p + (1-p)\{1+\log(1-p)/p\}^2$. Approximation error = Polylog$(2,p)-$ Approximate expression.



(b) Approximating the correlation by its average value over $(0,1)$: $c = -0.81$. $R_{\text{av}}(p) = 1.50377\{0.664995\,(1-p)\{\log(1-p)\}^2/p + (1-p)\{1+\log(1-p)/p\}^2\}$. Approximation error = Polylog$(2,p)-$ Approximate expression.

Figure 1.

Both representations for Polylog$(2,p)$ work well for p less than 40%, the error being very close to zero. Using the representation $R_{\text{av}}(p)$ permits accurate estimation of Spence's integral over a wider range: up to $p = 0.79$ (error $< 10\%$) or up to $p = 0.64$ (error $< 1\%$). In deriving these representations, we have approximated the correlation between X and $1/X$ by unity in one case, and by the average correlation over $(0,1)$ in the other case. These approximations have worked well in practice, and should therefore be of interest to practitioners.

Note that a family of approximations can be deduced from the expression for $R_c(p)$, each member of the family corresponding to a different choice of "c." Clearly, the lower bound of zero does not make sense since X and $1/X$ will always have nonzero correlation. I used the upper bound of unity (for the absolute value of the correlation) and the average value of $R_c(p)$ over $(0,1)$. Future research could examine other choices. In particular, if theory dictates a certain value of the correlation coefficient in a specific application, then that choice of "c" would be the most appropriate one for that application. In the absence of any such theoretical considerations, the average value seems an appropriate choice for "c."

CONCLUSIONS

An unexpected linkage between the second moment of the inverse of a geometric random variable and Spence's integral was shown to lead to simple representations for the integral. The representations work well over a reasonably large subset of the unit interval $(0,1)$ and should be useful to practitioners. The results provide a useful ballpark estimate of Spence's integral and are easily computable on a hand-held calculator. Apart from its practical significance, the results provide an interesting example of the intellectual interplay between the statistical and physical sciences.

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